Newtonian Gravity for the Non-Faint of Heart

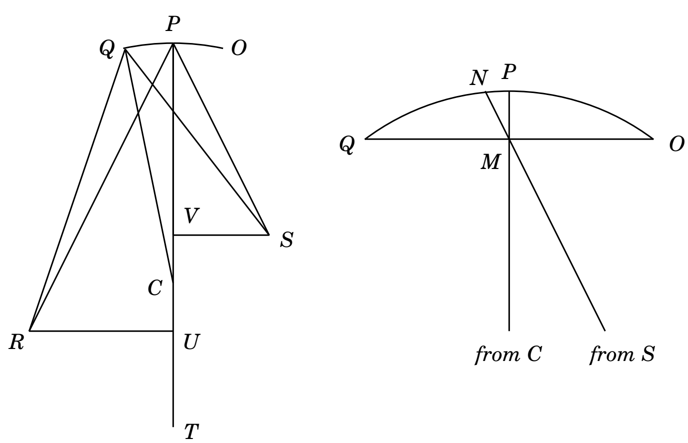
The following is a geometric demonstration of Proposition XI in Book I of Newton's *Principia Mathematica*—namely, that planets orbiting the Sun according to Kepler's laws must be subject to an acceleration inversely proportional to the square of their respective distances from the Sun.

It is not, however, Newton's own demonstration. I wanted to try to figure out a geometric demonstration on my own, without first reading Newton's or anyone else's, and after struggling with it off and on for several months, I was able to come up with one. I use a diagram that is strikingly similar to one in the Principia (down to the same similar triangles), but the argument, it turns out, is not really all that similar.

My demonstration (I hesitate to call it a proof, because I don't think it is quite rigorous enough) is in three parts, of which the first is the longest and most involved. It shows that a single planet in an elliptical orbit around the Sun in accordance with the law of areas is subject to a constantly varying acceleration, which always obeys the inverse square law.

The second part shows that all orbits with the same semi-major axis have the same proportionality constant in the inverse square law, thus tying all such orbits together—most notably circular orbits. Finally, the third part shows that all circular orbits that obey Kepler's third law must also share the same proportionality constant, thereby binding all orbits around the Sun to the same inverse square law.

In showing this relationship, I make implicit use (as Newton did himself) of his first two laws of motion, but not the third. If one does apply the third law, we find that not only are the planets subject to an acceleration toward the Sun, but because planets have mass, the Sun is in turn subject to an acceleration toward each of them—a fact that was of great conceptual importance and which is used today as the chief method of finding planets around other stars.



In the diagram above, the Sun is at S, the empty focus is at R, and the planet moves, in unit time, along the arc OPQ with $OP = PQ$. Knowing that the tangent of a small angle is approximately equal to the angle, and by considering similar triangles, we see that $\mathrm{m}\angle PRQ = (SP/RP)\,\mathrm{m}\angle PSQ$.

Bisect $\angle RPS$ with PT. Drop perpendiculars from R and S to PT at U and V, respectively. Also extend the bisector of $\angle RQS$ to intersect PT at C. The planet's orbit is perpendicular to CP at P, and to CQ at Q; hence, CP is the orbit's radius of curvature. Note that $\mathrm{m}\angle PCQ = (\mathrm{m}\angle PRQ + \mathrm{m}\angle PSQ)/2$, and our above discussion gives us

$$\mathrm{m}\angle PCQ = \frac{RP + SP}{2\,RP}\,\mathrm{m}\angle PSQ$$

Again using the small-angle approximation and similar triangles, we write

$$CP\,\mathrm{m}\angle PCQ = \left(\frac{SP}{VP}\right) SP\,\mathrm{m}\angle PSQ$$

$$CP\,\frac{RP + SP}{2\,RP} = \frac{SP^2}{VP}$$

which gives us

$$CP = \frac{2\,RP \cdot SP^2}{VP(RP + SP)}$$

By Kepler's second law, as the planet moves along the arc OPQ, it changes its velocity so that it sweeps out equal areas in equal times. Since M is the midpoint of the segment OQ, the areas of $\triangle OSM$ and $\triangle MSQ$ are equal, and hence the areas of the wedges OSN and NSQ very nearly equal. (To first order, they differ only by the small wedge PMN.) Therefore, the planet takes just as long to sweep from O to N as it does from N to Q, and the acceleration on the planet goes as MN.

For small arcs OPQ, $MQ^2 = MO \cdot MQ = MP(CM + CP) \doteq MP(2\,CP)$, and then

$$MP = \frac{MQ^2}{2\,CP}$$

By similar triangles,

$$MN = MP(SP/VP) = \frac{MQ^2 \cdot SP}{2\,CP \cdot VP}$$

From our determination of CP, we get

$$\begin{aligned} MN &= \frac{MQ^2 \cdot SP}{2\,VP} \cdot \frac{VP(RP + SP)}{2\,RP \cdot SP^2} \\ &= \frac{MQ^2(RP + SP)}{4\,RP \cdot SP} \end{aligned}$$

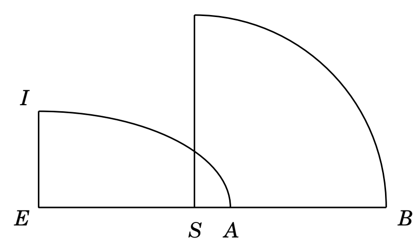
Again by Kepler's second law, $\mathrm{m}\angle PSQ$ goes as $1/SP^2$. MQ is very nearly $(SP/VP)SP\,\mathrm{m}\angle PSQ$, so MQ goes as $1/VP$. Since $RP + SP$ is fixed, MN goes as $1/(VP^2 \cdot RP \cdot SP)$, or as

$$\frac{1/SP^2}{RP(VP/SP)\cdot VP} = \frac{1/SP^2}{UP\cdot VP}$$

RS is constant, hence so is RS^2. By the law of cosines,

$$\begin{aligned} RS^2 &= RP^2 + SP^2 - 2\,RP\cdot SP\cos\angle RPS \\ &= RP^2 + SP^2 - 2\,RP\cdot SP(2\cos^2\angle VPS - 1) \\ &= RP^2 + 2\,RP\cdot SP + SP^2 - 4\,RP\cdot SP\cos^2\angle VPS \\ &= (RP+SP)^2 - 4(RP\cos\angle UPR)(SP\cos\angle VPS) \\ &= (RP+SP)^2 - 4\,UP\cdot VP \end{aligned}$$

Again, since $RP + SP$ is fixed, $UP \cdot VP$ must also be constant. Hence, the acceleration MN must go as $1/SP^2$—that is, inversely as the square of the distance SP of the planet from the Sun.



Now, suppose the center of the ellipse to be located at E, the Sun at S, the perihelion at A, and one end of the minor axis at I, as shown above. Then draw a circular orbit centered at S, with radius equal to EA. The radius of curvature of the ellipse at A is to that of the circle at B as EI^2 is to EA^2.

Once again by Kepler's second law, the velocity of a planet at A in the elliptical orbit is to that of the one at B in the circular orbit as EI is to SA, and thus the velocity squared as EI^2 is to SA^2. Combining these results, the acceleration of the planet at A in the elliptical orbit is to that of the one at B in the circular orbit as EA^2 or SB^2 is to SA^2, or equivalently as $1/SA^2$ is to $1/SB^2$—that is, inversely as the square of their distances from the Sun.

Finally, suppose two planets to travel in circular orbits, centered on S, whose radii are as a^2 is to 1. Then by Kepler's third law, their periods are as a^3 is to 1, and thus their velocities as 1 is to a. Hence, their accelerations are as $1/a^2$ is to a^2, or equivalently as $1/a^4$ is to 1—that is, again, inversely as the square of their distances from the Sun.